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Mathematical Modeling: Cognitive, Pedagogical, Historical And Political Dimensions.

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> "When I mentioned to my family that I intended to write a book about music there was a moment's silence and then laughter. Although I frequently listen to music, I can neither sing in tune nor clap any rhythm. I am unable to play any musical instrument."
>
> Steven Mithen, 2006.

Abstract
The book by Steven Mithen became a basic reference on human evolution. The introductory words of Mithen encouraged me to write a paper on Mathematical Modeling. Although having practiced Mathematical Modeling only with situations which are not complex, I believe my paper contributes to the foundations of the field. In the late sixties I became interested in the emerging field of Mathematical Modeling. I practiced some easy modeling and I supervised a number of dissertations, dealing with more complex situations. This was an exciting experience and I immersed myself in the relevant literature of the field. Although I never got interested in practicing more advanced modeling, the emergence of new fields of research after the sixties convinced me of the central importance of Mathematical Modeling in explaining knowledge. The objective of this paper is to show some of this centrality as I see it. Particularly, I will try to explain why I do recognize Mathematical Modeling as a strategy per excellence of human beings for generating knowledge and for learning. My exposition is based on assuming some concepts. First of all, what is knowledge? Knowledge is one of these concepts for which we do not have a definition and philosophers take for granted that the reader will understand what they mean when they refer to knowledge. I do the same. But I distinguish <u>individual knowledge</u> and general, or simply, <u>knowledge</u>. Individual knowledge is generated by each individual. Socialization leads individuals to acquiring and learning knowledge generated by other individuals, and this process is in a symbiotic relation with the continuous generation of new knowledge by the individual. Socialization depends on communication in the broad sense. Language, as the main form of communication of the human species, allows for sharing, transmitting and diffusing the complex of experiences and explanations accumulated by mankind in the course of its history, since early hominids through the modern *homo sapiens sapiens*. This leads to general knowledge, or simply <u>knowledge</u>.

Keywords: Mathematical Modeling, Knowledge, Mathematics Education, Problem Solving

1. Knowledge

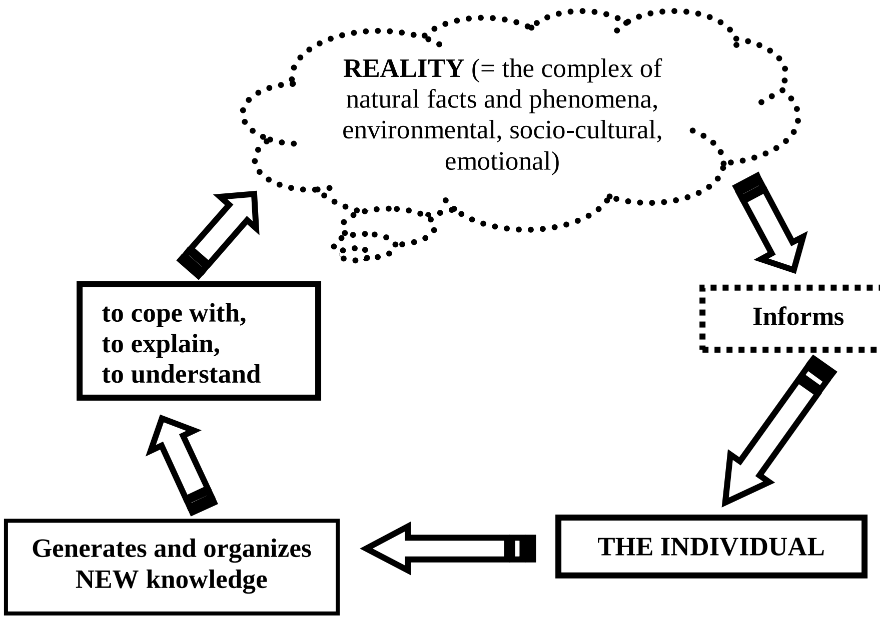
By individual knowledge I mean the complex of experiences and explanations accumulated by an individual, along his/her life, as an individual response to his/her needs and will. Knowledge determines behavior. For an individual, knowing and doing are not a dichotomy.

How is individual knowledge generated and how is it intellectually organized by the individual, as a response to his/her needs and will? These are open questions in philosophy. I just assume that individuals generate and organize knowledge, in a cyclic process, as a result of stimuli from reality. Reality is understood as the complex of interrelated natural, environmental, socio-cultural, emotional facts and phenomena. Reality informs the individual providing stimulus for action. The individual receives information by the senses and individual and genetic (DNA) memory. The individual processes the information by neuro-physiological mechanisms and proceeds to action, spontaneously (as for example, as a response to instinct) or following a strategy of action, with specific objectives. Receiving and processing information are, as yet, not fully explained. Probably will never be, but models that simulate these processes are known. The rapidly developing field of artificial intelligence contributes to this. The result of processing information is action. Examples of action are behavior and the generation of knowledge. Action has its effects in reality, modifying it. Hence, reality is continuously modified and, consequently, the individual always receives new information.

As a consequence, action is always modified. Hence behavior and knowledge are permanently transformed. This justifies the image of a cyclic dynamics of the generation of knowledge by the individual, as below:

••• □ reality □ individual □ action □ reality □ individual □ action •••

THE INDIVIDUAL GENERATION OF KNOWLEDGE



This image synthesizes the main ideas which I developed since the late seventies (D'Ambrosio 1992).

In all animal species, except the species *homo*, the processed information leads to instinctive action, in response to the satisfaction of needs. In the species *homo*, the processed information results in strategies for action which subordinate instinct and the satisfaction of needs to will. This explains continence, refusal to follow orders and norms and, even, suicide.

The individual is not alone. Gregariousness is a characteristic of animal species. How do an individual and others interact? Communication plays a fundamental role in the interaction. Thanks to communication in the broad sense, every individual share knowledge with other individuals (kin, community, society).

Communication among humans is a most intriguing issue. Particularly when it focus on myths and social actions, acquiring the sophistication of music and language, responsible for sharing believes and knowledge (Christiansen 2003). The evolution of communication into language is well illustrated in the movie *Quest for Fire*, of 1981, directed by Jean-Jacques Annaud.

To share individual knowledge, which includes beliefs, with other individuals, with different needs and will, is the great challenge to social life. Beliefs are responses to needs and will of an individual are not necessarily coincident with beliefs of others. Hence, behavior, and even knowledge, may be different for each individual of the group. This gives origins to conflicts within the group. This is the reason why there are, in every society, outcasts. To deal with outcasts, every society proposes punishment as correctional measures and education as preventive measures. A very interesting illustration of this is the theme of the classic book of 1962, *A Clockwork Orange*, by Anthony Burgess

An important step in the social evolution of the species, which may be explained as a response to gregariousness, is to make compatible the different individual responses of the group. The parameters guiding compatibility are systems of values, which have their origin in myths which are common to the individual of the group. How do they become common knowledge? This is a very difficult question. It is related with the origins of symbols and the emergence of the sacred as an answer to the inexplicable, which are, both, the substratum of languages, arts and the sacred.

Once succeeding in achieving compatibility, knowledge, subjected to systems of values, is shared by the group. This defines the culture of the group: shared knowledge, compatible behavior, accorded system of values. Knowledge shared by the group is socially organized, thus becoming a body of knowledge, which is a response of the needs and will of the individuals of the group.

Needs and will are characteristics of the species *homo* and are the response of the pulsion of survival, present in every animal species, and the pulsion of transcendence, present only in the human species.

In the search of survival, the species develop behavior and knowledge that are necessary for the survival of the individuals and of the species, basically the means to cope with the most immediate environment, which supplies air, water, food and the recognition of a different individual (male/female) necessary for procreation. These strategies are modes of behavior and individual and collective knowledge, which include communication and, in the species *homo*, language and the creation of instruments.

In the search of transcendence, the human species develop the perception of past, present and future, and their linkage, and the explanations of facts and phenomena encountered in their natural and imaginary environment. These are incorporated to the memory, individual and collective, and organized as arts, techniques, which evolve as representations of the real [models], the elaboration about these representations, the organization of systems about explanations of the origins and the creation of myths and mysteries. Attempts to know the future [divinatory arts] encounter support in the memory, where we find the myths, the mysteries, history and the traditions. The collective responses to mysteries and myths are organized as languages, arts, among them the divination arts, the sacred and religions and value systems. This is a basic question in explaining religion, as for example in Otte 1993. These organized responses give origin to systems of knowledge, like astrology, the oracles, logic, the *I Ching*, numerology and the sciences, in general, through which we may know what will happen – before it happens! Systems of knowledge provide explanations to the myths and mysteries and link past-present-future.

This leads to a definition:

Models are representations of the real and modeling is the elaboration about these representations, particularly how these representations are created and how one can, by elaborating about the models, can draw information about the real.

I believe the considerations above justify recognizing modeling, as just defined, as the strategy per excellence of human beings for generating knowledge.

2. Mathematical Modeling

The practice of creating models and elaborating on them utilizes material and intellectual instruments. In the case of the utilization of mathematical instruments, we refer to this practice as Mathematical Modeling.

The objectives of Mathematical Modeling are multiple. Observations on the behavior of a model may serve as a guide for experimental tests. Besides, models can help to identify and to ask new questions.

Since models provide only approximations of the real behavior, they also may help reformulate hypotheses and even to formulate new ones, laying the ground for new theories, more appropriate to deal with the original question. Thus, models may be more or less useful, and are subjected to reformulations, leading to better approximations.

Mathematical Modeling became an important instrument in Business, Administration, Health Sciences, Environment, indeed, in every human activity.

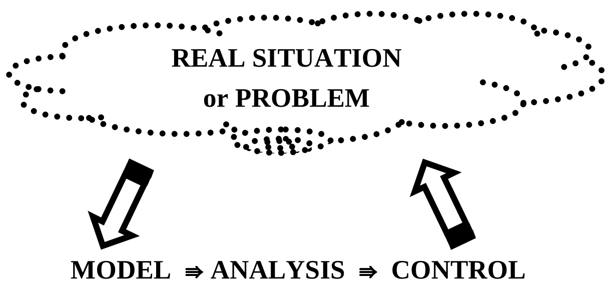
For example, a cathedral is a model of an ambiance for worship -- rooted in mystical natural environments – realized as a building with specific architectural elements, is a model of artistic nature. A geometric theory, for example, Euclidean geometry is a theoretical model of forms and relations among the components of these forms. Cosmic system, specifically the solar system, was modeled by Ptolemy, in the 1st century. It was extremely important and useful. The great Portuguese discoveries in the 15th century, were based on the Ptolemaic model. But as new observations of the real system are available, new elements are being introduced in the representation and a more refined model is proposed. Thus it was possible for Copernicus to make another model of the solar system, at the beginning of the 16th century.

The utilization of mathematical models of reality depends on the knowledge of facts and phenomena, of the recognizable behavior of the real objects and systems, normally expressed by laws, mostly derived empirically. To deal and draw benefits from mathematical models, intellectual tools are provided by mathematics, considered a body of concepts and theories and the operations rules to deal with them. The effectiveness of these tools is well expressed by the eminent physicist Eugene Wigner:

> "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve"(1960).

Although recognizing that an ultimate truth is unattainable, better approximations are achieved with Mathematical Modeling. Another advantage of Mathematical Modeling is to validate and to make predictions about the behavior of the system and try to control the system. With the development of computing, the Mathematical Modeling, particularly simulation, has become, probably, the most powerful scientific tool we have. The mathematician Jacques-Louis Lions did an excellent study on the possibilities of Mathematical Modeling to study the planet Earth (1960).

In this book, Lions introduces the concept of the "**universal trilogy**"



which clarifies the image of the individual cycle of knowledge, presented above.

This leads to the most difficult and the most ambitious proposal of man's intervention in natural processes.

To conceptualize interventionist procedures, we refer to the above image. Once a very complex real situation or problem is recognized, such as those of biology and the environment, the individual identifies a number of variables and proposes a model. The choice of variables determines how accurate is the model, that is, the level of approximation to the real situation or problem. Thanks to new material and intellectual instruments, the individual identifies more variables, which allows for more accurate models. History of Science has been exactly this: to improve theories, based on models of reality, thanks to the development of new intellectual and material instruments. The evolution of Ptolemy→Copernicus→Galileo systems, of Newtonian to Quantum Mechanics, and also molecular biology clearly illustrate the effects of new intellectual and material instruments in the History of Sciences. Once in possession of models, thanks to theoretical instruments, mainly provided by mathematics, it is possible to understand and analyze the model and to create techniques of control. Again, the important remarks of Eugene Wigner cited above clearly support these views. These, applied to the real situation or problem, may interfere with the course of nature.

To apply the possibilities of interventions in natural and social process, thanks to the "universal trilogy", is the domain of political decision. This leads, for example, to strategies for environmental and health policies. For example, the control of emissions and the inclusion of genetic medicine in primary health care. The interventions are a form of generating and organizing new knowledge about the real situation and working, analytically, on the model, and as the cyclic nature of the process determines, constantly control the results. Both images explain the process cognition.

3. Mathematics Education

Mathematical Modeling is an important component of professional training, which is very much alike in all areas, particularly in Mathematics Education.

The incorporation of Mathematics Modeling in Mathematics Education leads, indeed, to creating a learning environment. This requires a new approach to the objectives teaching mathematics. Two major obstacles are that the situations are normally open ended or may offer multiple options and that the students are not asked for individual solutions. The learning environment thus created is a form of "open classroom" practice. Students have to work together.

One of the difficulties of working together is, both in classroom and professional environments, that the parties often have different perceptions of the same situation. The challenge is to explore the dynamics of different perceptions and strategies, turning them into a group strategy. The UNESCO proposal of Cooperative Problem Solving may be easily adapted for an in-service course on Mathematical Modeling for teachers (UNESCO 2003).

There are obvious relations of modeling with problem posing + solving, representative of a new thinking in Mathematics Education. It calls for situations involving several variables, and an important role is played by selecting the variables which can be handled by the modeler. To create the model and to elaborate on the model is, thus, limited by the mathematical instruments available to the modeler. As a cooperative effort, modeling is a group effort, hence more instruments can be available, which will result in a larger selection of variables and, consequently, in better approximation to the real situation.

In the late seventies and early eighties, we benefited, at UNICAMP, of yearly visits of Hassler Whitney. He was very influential in the development of a line of research in Mathematics Education, which is incorporated in the Program Ethnomathematics. In a very inspiring writing, a pre-publication of a book which never came to formal printing, Hassler Whitney gives the basic ideas of modeling when proposing the following reflection on story problems, which are, indeed, situations:

> "How does one solve story problems? First of all, replace 'solve it' by 'play around with it'. Half the difficulty is now over. Make it concrete: act out the story. Have courage to try the story in different ways, getting used to its various features. When things turn out wrong, be in interested in how they are wrong, and try changes; act out the story again. Now if you ask what was wanted, you may be ready to see or quickly find the answer. It is really basically as simple as that. Courage to play with and try different things is the key note."(1976 , p. 41)

I feel current research in Mathematics Education is not giving enough attention to the nature of mathematical reasoning. Consciousness Studies, or the Sciences of the Mind, are not regarded as valuable to research in Mathematics Education. I even notice some contempt about these research areas. There is an insistence on sometimes obsolete cognitive science. Contributions of Gregory Bateson, Francisco Varela, Humberto Maturana are, in general, ignored. Their proposals lead, naturally, to modeling.

Again, I give the opinion of Hassler Whitney, himself a very creative and original mathematician, about playing with story problems, which are, as mentioned above, situations:

> "The child's need is to live and grow in his own way. Story problems become part of oneself when acted out; in this way, they become real, and in particular, numerical relations appear naturally.
> The child's natural approach is to experiment and explore. Carrying this out, he finds courage to try many things; some work out in funny ways (which we prefer to call wrong), others come out right. Intrigued, he plays with the funny things, changes them, sees what happens, and makes them come out right also. He is beginning to act like a research worker."(Whitney 1974, p.3)

What we may need is a sort of re-conceptualizing the relations teacher-student. To give voice to the student when faced with a challenging situation and to listen becomes more important than to teach students' how to find solutions to a problem. To create learning experiences as the result of children playing with modeled real situations illustrates classroom research, as, for example, is shown by Beatriz D'Ambrosio (2004).

4. Modeling fictitious situations may lead to mathematical problems

Modeling of fictitious situations which resemble real situations may lead, in a very natural way, to challenging mathematical problems. Challenging mathematical problems have been, historically, impelling forces in the development of new mathematical theories.

We may analyze the advancement of mathematics in the general framework of historiography, which recognizes that *ad hoc* dealing with situations and problems leads to the generation and organization of knowledge. Organized knowledge is what we call a method.

As discussed above, the initial step of recognizing a situation or a problem leads to modeling the situation or problem. The next step is the work on the model.

Working mathematically with the model goes beyond the application of a sequence of algorithms. Working with the model is not only applying the linked procedures of induction and deduction, as normally presented.

A model represents a situation or problem, generally new. Mathematics is the most powerful instrument created by humans to deal with new situations, to face unpredicted problems. I believe Mathematics Education has to convey this. The basic, fundamental step, for creating these mathematical instruments to deal with a new situation and to face an unpredicted problem is the process of abduction, as proposed by Charles Sanders Peirce. Basically, it is guessing, trying, reformulating. All these are present in Mathematical Modeling.

The next step following *ad hoc* solutions is the organization of a method. Only from methods we can proceed to theories and from there to invention. This explains the History of Mathematics.

Indeed, in Western Mathematics, problems are present in Egyptian and Babylonian sources. Euclid's *Data*, which we may consider the pedagogical strand of Euclid, is a collection of problems. And in the Middle Ages, challenging problems were famous. For example, in the *Tractatus Algorismi*, of Jacopo da Firenze (Montpelier 1307), we read:

> "I go to a garden, and come to the foot of an orange. And I pick one of them. And then I pick the tenth of the remainder. Then comes another after me, and picks two of them, and again the tenth of the remainder. Then comes another and picks three of them, and again the tenth of the remainder. . . . And then come many. Then the one who comes last, that is, behind, picks all that which he finds left. And finds by this neither more nor less than the others got. And one picked as much as the other. And as many men as there were, so many oranges each one got. I want to know how many men there were, and how many oranges they picked (each) one, and how many they picked all together." (Høyrup , 2007, p. 6)

This exemplifies modeling a situation. This kind of challenge was responsible for the enormous creativity which paved the way to Renaissance and Modern Science. But, at the same time, the author searches for a formula to solve such problem, and, once he learns of a formula, he begins to teach it. The formula dominates the problem. To teach the formula becomes the essence of the education of a mathematician. The student may master the instruments, may even find answers, but the understanding of the problem will be deficient.

The reaction to this is probably what is in the background of Descartes' discourse in the explanatory paragraph of his *Discourse of Method*, published in 1637, who set the basic ideas of Modeling, synthesized in four steps:

> "I believed that I should find the four [steps] which I shall state quite sufficient, provided that I adhered to a firm and constant resolve never on any single occasion to fail in their observance.

> The first of these was to accept nothing as true which I did not clearly recognize to be so: that is to say, carefully to avoid precipitation and prejudice in judgments, and to accept in them nothing more than what was presented to my mind so clearly and distinctly that I could have no occasion to doubt it;
> The second was to divide up each of the difficulties which I examined into as many parts as possible, and as seemed requisite in order that it might be resolved in the best manner possible;
> The third was to carry on my reflections in due order, commencing with objects that were the most simple and easy to understand, in order to rise little by little, or by degree, to knowledge of the most complex, assuming an order, even if a fictitious one, among those which do not follow a natural sequence relatively to one another;
> The last was in all cases to make enumerations so complete and reviews so general that I should be certain of having omitted nothing."(Descartes 1994, p. 271)

Although these are well known, mathematics educators do not give enough attention to the following remarks, preceding the listing of the four steps of his method:

> "I had, in my younger days to a certain extent studied Logic; and in those of Mathematics, Geometrical Analysis and Algebra – three arts or sciences which seemed as though they ought to contribute something to the design I had in view. But in examining them I observed in respect to Logic that the syllogisms and the greater part of the other teaching served better in explaining to others those things that one knows . . . than in learning what is new. . . . And as to the Analysis of the ancients and the Algebra of the moderns, besides the fact that they embrace only matters the most abstract, such as appears to have no actual use, the former is always so restricted to the considerations of symbols that it cannot exercise the Understanding without greatly fatiguing the Imagination; and in the latter one is so subjected to certain rules and formulas that the result is the construction of an art which is confused and obscure, and which embarrasses the mind, instead of a science which contributes to its cultivation. This made me feel that some other Method must be found, which, comprising the advantages of the three, is yet exempt from their faults. And as a multiplicity of laws often furnishes excuses for evil-doing, and as a State is hence much better ruled when, having but very few laws, these are most strictly observed; so, instead of the great number of precepts of which Logic is composed."(Descartes 1994, p. 270f)

Two phrases of Descartes are of high actuality: "explaining to others those things that one knows **. . .** than in learning what is new" and "the considerations of symbols that it cannot exercise the Understanding without greatly fatiguing the Imagination".

These are the crux of the perceived lack of success in Mathematics Education. There is not space for Creativity. The Method Descartes proposes is the search for the new. It is a sort of guideline for research. Mathematical Modeling must be interpreted as research, with the goals of finding the new, and not with the goals of repeating what is well known.

The educational experience is the result of dynamics on the encounter of individuals with different knowledge. Descartes does not teach his method, he tells about it to the reader. He tells about four steps he uses to find the new. This writing reveals a form of dynamics of the encounter of Descartes' culture and the reader's culture. Learning and new knowledge are the result of this encounter. Both, teacher and students, are engaged in the encounter, a sort of play, in which the culture of one player/teacher (his intention) and of the other player/student (his curiosity) give rise to a new culture, shared by both players. This is the meaning of creativity in the educational process. Indeed, we are again proposing cooperative learning, as already discussed above.

Regrettably, the emergence of symbolic formalism, and of formulae, more attention is given to form than to substance, which interfered with creativity. From Descartes' explanatory paragraph, cited above, this interference was his motivation for his proposal of a Method.

5. Problem Solving and Mathematical Modeling and the equivocal practice of testing

Historically, we see Problem Solving present, as a pedagogical practice, in just about every mathematics curriculum. It is practiced, in general, to illustrate theoretical treatment and the focus is to obtain correct answers.

Another approach to Problem Solving, which is a good pedagogical practice, is to get started with problems to prepare the ground for theoretical treatment. The classic *Elements of Algebra*, by Leonhard Euler, is very illustrative of the use of problems to convey more general and abstract ideas. Although the book starts with problems which are easy to solve, this leads to theory, which opens to way to creative and challenging problems.

Indeed, challenging problems, of which the elementary question of Jacopo de Firenze, given above, is a good example, have always been a basic motivation for mathematics research. The collection of problems in the famous *Scottish Book*, held in the Scottish Café, in Lvov, Poland, in the thirties, played a very important role in the development of mathematics in the pre-WW II period. Mathematicians from all over the World would visit the Scottish Café to learn about new challenging problems. The idea of offering a Problem Section is still common in some scholarly journals. These are harder and require deeper insight. These problems attract prospective mathematicians. Hungary has been a pioneer in these contests. The Eötvos Competition has been, since the beginning of the 20^{th} century, an educational strategy and Georg Pólya was active in these contests. Now, the Olympiads of Mathematics follow this tradition. As I discussed above, challenging problems are the initial steps for new theories. It is common to speak of "problems posed by life". These clearly call for Mathematical Modeling.

There is a danger in restricting the practice of Mathematical Modeling to problems posed by life. Fictitious, imaginary situations and fantasy are equally important. We may use fiction as a resource to create a situation which have elements of real life mixed with elements of fiction. This I call a "fictitious real situation". The following phrase attributed to Sophus Lie, the distinguished Norwegian mathematician, in a letter written to Bjornson, in 1893, is very appropriate:

> "Without Fantasy one would never become a mathematician, and what gave me a place among the mathematicians of our day, despite my lack of knowledge and form, was the audacity of my thinking (Stubhaug, A. S. 2000, p.143)."

A number of tales of the classic *The Thousand and One Nights* are mathematical problems posed by "fictitious real life" situations. It is interesting to mention Júlio Cesar de Mello e Souza (1895-1974), a Brazilian mathematician and mathematics educator, who published a number of text books for elementary, secondary and higher education. Under the pseudonym of Malba Tahan, he published books of tales inspired by *The Thousand and One Nights*. His most successful book, published in 1949, is *O Homem que Calculava* [*The Man Who Counted*], whose main character, Beremiz Samir, was a sophisticated Problem Solver. The book discusses highly challenging problems, dealing with the most diverse situations of "fictional real" life. The book was translated into several languages. Most of the problems require creating mathematical models which capture the imagination of children and adults alike, and require mathematical creativeness to a degree generally not contemplated in School Mathematics (Tahan, M. 1993).

The Program Ethnomathematics contemplates, inherently, Mathematical Modeling. It discusses the evolution of mathematical knowledge, both in the history of an individual and in the history of the human species, as a response to a variety of situations and problems, of different nature.

This is formalized in the three basic steps of research:

1. How to pass from *ad hoc* solutions and practices to methods;
2. How to from methods to theories;
3. How to pass from theories to invention.

As discussed above, the individual, when facing a new situation or a problem, tries to understand or to solve it. This is an *ad hoc* solution. The next step is to try the same procedure to similar situations and problems. In the case of succeeding, this is repeated, and a method has been achieved. This allows for creating new ways of dealing with new situations and problems. These steps are the essence of Mathematical Modeling.

A major obstacle to introduce Mathematical Modeling in mathematics curricula may be the emphasis in testing. I am convinced that testing, both national and international, is damaging creative problem solving. There are no possibility of significant progress in Problem Solving and in Mathematical Modeling if current practices of testing, both national and international, prevail. The results of testing determine the financing of school systems and even the career of teachers.

There is enough evidence that assessment affects teachers and their practice. Since teachers' are supposed, and indeed required to prepare students to do better in the tests, there is not the necessary freedom for the teacher to innovate. Teachers have to act more like trainers than as educators.

An innovative approach of Problem Solving, and the same is true for Mathematical Modeling, depends, paraphrasing Hassler Whitney, on the courage to present complex situations and hard problems, some even unsolvable, to students, and, instead of asking for solutions, to listen to their proposals and to give space for their creativity to manifest.

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