Inconceivable magnitude estimation problems: an opportunity to introduce modelling in Secondary School

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Abstract
Fermi problems are problems which, due to their difficulty, can be satisfactorily solved by being broken down into smaller pieces that are solved separately. In this article, we present inconceivable magnitude estimation problems as a subgroup of Fermi problems. Based on data collected from a study carried out with 12 to 16-year-old students, we study secondary school students' resolution proposals for inconceivable magnitude estimation problems. We analyze their suggestions with the aim to characterize their strategies and detect the modelling processes involved. We have detected several different types of models suggested (iteration of a reference point, concentration measurement, stratification of a population or rows and columns distribution) by the students and we conclude that secondary school students have enough mathematical knowledge to address problems involving inconceivable magnitude estimation and can attempt to solve them by employing modelling processes.

Keywords: mathematical modelling, compulsory secondary education, estimation, problem solving.

1 Introduction
In recent decades, problem-solving has been widely researched within Mathematics Education (Lester 1994, Schoenfeld 2007). Puig (2008) provides a view on the state of problem-solving in Spain which emphasizes the gap between research and its contributions to curricular development in the last few years. Thus, problem-solving has been present in the curricula but its use hasn't been fully established in the classrooms. However, modelling is considered a very valuable scientific activity, which should be included in educative curricula in order to promote the development of the students' mathematical competences.

The stage of Compulsory Secondary Education (ESO or Educación Secundaria Obligatoria) in Spain is comprised of 4 academic years for students aged 12 to 16. Introducing mathematical modelling in secondary school initially gives rise to potential difficulties: Are the students competent enough at problem-solving? Are they limited by secondary school's mathematics curriculum? Are they able to generate mathematical models by themselves? For what kind of situations? For these reasons, our study is based on this level of education.

In this article we put forward a set of problems which involve estimating inconceivable magnitudes. These problems have been used as an educational resource to introduce modelling in mathematics classrooms within the context of problem-solving, in order to overcome the aforementioned difficulties. In addition, this kind of problems aim to cover several needs at the same time; on the one hand by suggesting problems based on real situations and, on the other, by promoting estimation and the development of mathematical models.

2 Theoretical references

2.1 Problem-solving

Numerous research studies in the field of Mathematics Education deal with problem-solving. One of the earliest works on the subject, Pólya (1945), establishes a four-stage model by which a problem can
be solved. These are the following: i) Understanding the problem; ii) Devising a plan; iii) Carrying out then plan; iv) Looking back.

Pólya uses introspection to examine a so-called ideal problem solver's behaviour, that is, a solver who is capable of self-managing his own resolution task and follows the aforementioned four stages lineally, moving on to the next phase only when the previous one has been successfully completed. The model presented by Pólya also includes a series of questions which the solver can ask himself in order to make progress in his task. The great majority of these questions are variations of ¿do you know any related problems? and this thus seems to be the driving force of problem-solving according to Pólya. Several authors have expanded Pólya's suggested problem-solving model, however keeping the essence of a four-stage structure.

The first stage consists of the identification, definition and understanding of the problem. In this phase, the problem solver recognizes the existence of a problem and the need to solve it. The second stage is focused on planning the resolution. This includes designing the action scheme and identifying the objectives to achieve after examining the general strategies which could be applied. The third stage is the execution of the previously designed plan and the fourth one involves verifying the task and decisions made, as well as validating the resolution and results obtained from the original plan.

Given that this study deals with problems which a teacher poses to students in a school environment, we adopt Puig (1996)'s definition of problem.

An educational mathematics problem is a task with mathematical content, the formulation of which is significant to the pupil who is presented with it, wishes to approach it, and has not yet made any sense of it. (page 31).

In our study we focus on the resolution of problems with formulations established within a real-life context. Following Winter (1994), solving problems with a real context involves mathematizing a non-mathematic situation—which means constructing a mathematical model that will respect the real setting—calculating the solution and transferring the obtained result from the model to the real-life scenario. The most difficult step in this process is establishing an appropriate model for the suggested true situation, since it requires good knowledge of the context and a great deal of creativity, amongst other things.

The issues we raise with the pupils in our study could specifically be considered Fermi problems. Årlebäck (2009)'s suggested definition of Fermi problem is the following:

Open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations (page 331).

An argument for the use of Fermi problems in the classroom is the possibility of using them as a bridge between mathematics and other school subjects, approaching the students to different interdisciplinary tasks, as justified by Sriraman & Lesh (2006). In fact, Peter-Koop (2004) puts forward that Fermi problems are better and more useful when not considered as merely intellectual exercises and are focused on real situations and contexts within our everyday environment.

2.2. Modelling

One of the most relevant scientific activities nowadays involves creating models which may recreate abstract-shaped objects, processes or phenomena which we set out to understand to a great extent. Modelling situations generally aim to focus on the ability to predict, simulate or obtain accurate-enough descriptions of the studied reality.

Producing models to solve problems is not unique to the highest scientific levels, it has actually been recorded in several studies within the field of education. Lesh & Harel (2003) state that “the products that problem solvers produce generally involve much more than simply giving brief answers to well formulated questions” (page 158). In fact, in recent years there has been a tendency to try and bring model creation to the classrooms.
If we focus on modelling in the field of Mathematics Education, Lesh & Harel (2003) define the concept of model as follows:

Models are conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience-based metaphors. Their purposes are to construct, describe or explain other system(s). Models include both: (a) a conceptual system for describing or explaining the relevant mathematical objects, relations, actions, patterns, and regularities that are attributed to the problem-solving situation; and (b) accompanying procedures for generating useful constructions, manipulations, or predictions for achieving clearly recognized goals (page 159).

Modelling a phenomenon requires connecting mathematical concepts and operations with reality, giving meaning to the studied subject due to the need to symbolically describe a situation (Lesh & Zawojewski 2007), while generating models which are truly applicable to a given reality and can be generalized, being able to interpret the solutions derived from them (English 2006, Doerr & English 2003).

Several different views exist on the way students elaborate models for solving problems, and are a matter of discussion (Borromeo Ferri 2006). However, it has generally been accepted as a multi-cyclical process. This means that the students go from the real situation to the model and from the solution offered by the model to a solution for the problem by means of a series of processes they are constantly revising.

Following Blum (2003), modelling processes can be structured into five main stages: i) Simplifying the real problem into a real model; ii) Mathematizing the real model into a mathematical model; iii) Searching for a solution from the mathematical model; iv) Interpreting the solution of the mathematical model and v) Validating the solution within the context of the real-life problem.

These stages are combined in the following diagram by Blum & Leiss (2007):

![Blum & Leiss (2007)' cycle of modelling/ modelling cycle](image)

These processes are not always easy for students, some specific difficulties have been recorded such as the presence of too many lineal models in situations which do not require them (Esteley, Villarreal & Alagia 2010).

3. Methodology

3.1. Problems used in our study

Given that the objective of this study is to consider the possibilities of introducing modelling in secondary school, it may be useful to present the students with problems suggesting situations which they haven't previously studied but which are familiar to them. In order to encourage the need to use models, we propose problems which are centered on estimating inconceivable amounts, since this will force the students to use a method for simplifying reality and to mathematize it.
We say we make an estimate when we try to answer questions such as: how long am I going to take to get to work? How many 5 kg cans of paint do I need to paint the whole dining room? Or, how many tablespoons of oil do I need to reach the 30 grams mentioned in the recipe? Segovia, Castro, Castro & Rico (1989) define mathematical estimation as the judgment of the value which anticipates the result of a numerical operation or the measurement of an amount as a function of the individual circumstances of the person calculating it.

Many are the questions we could pose which would find a valid answer in numerical estimation. The amount of paint we need to paint the dining room or the time needed to get to our workplace are difficult quantities to determine accurately. In these cases it is more efficient to find an approximate solution than to aim to calculate the exact solution. However, we must bear in mind that all the necessary data won't always be available to us, or we may not have enough time or the required knowledge to come up with an answer. In fact, some of these questions do not allow for what we understand as a solution in the strictest sense, since its final value can be influenced by different factors which are determined by the way the situation is specified, how the problem has been set out and which direction has been chosen for approaching it.

In this study we have focused on discrete quantities but continuous magnitudes may appear in the resolution process. The suggested problems can be divided into simpler problems which can be resolved separately they can therefore be considered Fermi problems applied to inconceivable magnitude estimation.

Some of the problems that fit into our scheme may appear to be anecdotic, such as asking ourselves how long it would take to cycle from Paris to Beijing. However, the content of other problems is helpful for understanding the environment and is greatly socially relevant. Examples of the latter would be the estimation of the amount of people in a demonstration, the amount of water consumption or the amount of litter generated in a town. Discussing the social matters which appear in these problems in class may help in the student's knowledge of his/her environment as well as to depict a familiar situation on which to set out a problem that belongs to their own reality and on which they can develop their critical thinking.

3.2. Our study

In this study we analyze the possibilities of introducing modelling to the curriculum and promoting estimation and problem-solving through the use of Fermi problems applied to inconceivable magnitude estimation. In order to obtain a complete description of the students' resolution of these problems we should consider the understanding of the problem, devising a plan, its execution and its review. However, we have limited our study to analyzing the students' resolution proposals. We therefore limit ourselves to present students with problems only asking them to give us an action plan in order to solve them.

We begin with a list of 36 problems with formulations which are set out in different contexts and pose questions involving different kinds of discrete and continuous magnitudes. Some examples of the problems included in this first list are: the estimation of the time needed to copy out a book by hand, the number of hairs on somebody's head, the number of cars which drive along a certain road or the joint value of all the cars of a town. Some of these problems are original, while others are variations or adaptations of those found in one of the following lists:


• Classic Fermi Questions with annotated solutions at the website of the Collin County Community College District. Last visited/checked: July-2009. [http://iws.ccccd.edu/mbrooks/demos/fermi_questions.htm](http://iws.ccccd.edu/mbrooks/demos/fermi_questions.htm)

The list we initially came up with contained rather heterogeneous types of problems. We decided to only include problems about discrete magnitudes, which involved counting the number of objects or people, since our pupils are more familiar with this kind of magnitudes. On one hand, the kinds of strategies needed to estimate continuous and discrete magnitudes may lead to very different and hardly comparable processes, adding difficulties to the analysis process. On the other hand, providing the students with just continuous magnitude problems would restrict the obtained data.

Sixteen of the thirty-six problems were used for a pilot test. For each of them, we created a worksheet containing a contextualized problem, asking the students to write a resolution plan for the question posed. Each worksheet was given to six students from different levels of secondary school. This trial enabled us to gather information to detect which problems would give us a wider collection of data. Using these data, we observed that when the problem's formulation called for the students to estimate the requested amounts, they would focus on reaching a solution and forget about writing out the action plan, so we decided the final question papers would explicitly demand only one draft of their resolution proposal. With respect to the selection of problems, we regarded those which the students could approach from a wider range of perspectives and which didn't give rise to errors in the situation set out due to possible ambiguities.

The following six problems were selected after the pilot test:

- **Problem A**: If we're planning to hold a concert in the high school playground, how many people could fit in it?
- **Problem B**: How many people are there in a demonstration?
- **Problem C**: How many text messages do people in your province send in one day?
- **Problem D**: There is a leak in the ceiling of the teacher staff room and a bucket has been placed beneath it. How many drops of water do we need to fill the bucket?
- **Problem E**: How many glasses of water do we require to fill a swimming pool?
- **Problem F**: How many one-euro coins fit in a cubic-meter safe?

Plainly, problems A and B refer to counting the people covering a surface area in two different contexts. At the same time, the premises suggested in problem A are known by the students, while in problem B people can occupy an undetermined surface and may get to move around. As far as problems D, E and F are concerned, they are centered on counting numbers of times a small volume will fit in a larger one. On the other hand, the enumeration demanded in Problem C requires some social knowledge of the environment.

These problems are presented to the students together with a formulation which contextualizes the question within a real situation. This formulation states that students should only write out one proposal for the resolution of the problem. For example, the following is the specific formulation for problem A:

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Mobile phones are useful for many different things (such as viewing pictures or videos, listening to music, playing games...), however, people still use them to communicate with each other, to chat online, to make phone calls and to send text messages. We are not usually aware of this, but these services require a huge telecommunications network. In this situation, a good question would be: how many text messages do all Catalans send in one day? Describe the steps you would follow to approximately calculate this amount using your own resources. You do not need to provide a solution, just explain how you would do it.
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We collected data from two secondary schools of the same town, a medium-sized city within the metropolitan area of Barcelona. One of these schools is a state school and the other one is private. These data were gathered in hour-long lessons, in which the students were able to make proposals for more than one problem. If a student finished the resolution proposal for the problem presented in their worksheet, the person in charge of collecting the data would offer them another one. In this way, we obtained 538 question sheets from 216 pupils. Table 1 shows the number of participating students for each academic level.
Inconceivable magnitude estimation problems: an opportunity...

Table 1: Number of participating students

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<th>Year</th>
<th>SCHOOL 1</th>
<th>SCHOOL 2</th>
<th>Total</th>
</tr>
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</tr>
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</tr>
<tr>
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</tr>
<tr>
<td></td>
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</table>

Table 2 shows the number of worksheets collected for the study from each year and type of problem.

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<tr>
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<th>PC</th>
<th>PD</th>
<th>PE</th>
<th>PF</th>
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<td>20</td>
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</tr>
<tr>
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<td>86</td>
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</tr>
</tbody>
</table>

Table 2: Number of worksheets collected per year and problem type.

4. Categories: Characterization of detected strategies

This analysis was developed using NVivo 8, a qualitative data analysis software package which works on data digitalization, codification and categorization, enabling a quick and handy management of data and categories, facilitating a dynamic relational analysis. With the help of this program we have characterized the strategies suggested by the students for each problem, observing which of these contained elements of modelling.

As follows, we present the characterization of the strategies suggested by the students for each problem, analyzing whether they include modelling elements or not. Since the length of this paper doesn't allow us to provide details on the analysis process, the categories are illustrated by examples from the students' answers in order to make the process more intelligible. The student's proposals have been translated from Catalan trying to respect the structures and expressions the students used. Each proposal is listed together with the problem it corresponds to and the student's level of studies.

The answers on the worksheets have been differentiated into three categories, regarding their modelling content:

• Those which do not include any kind of resolution strategy
• Those which contain resolution strategies without elements of modelling
• Those which contain resolution strategies with elements of modelling

4.1. Proposals without a strategy

Firstly, we shall illustrate the kind of answers we call proposals without a strategy. This category includes those proposals which do not attempt to solve the problem set out and those which display a series of unconnected or incoherent ideas.

This is depicted in the following proposal by a 2nd of ESO pupil for problem D:

**PD2 - leakage**

*It depends on the size of each drop whether they will fall into the bucket by splashing out small drops or not, and also on the size of the bucket. Mind the drops don't evaporate.*
In this case, we can observe that this student has the correct idea on the kind of problem presented in this context, identifying some of the elements required for its resolution. He is able to see a relationship between the size of the bucket and the size of the drops, but cannot properly explain it. At the same time, the complexity of the situation causes less relevant factors like the water's evaporation rate to interfere with his proposal. He hands in his worksheet without specifying any steps to try and give an answer to the question posed, so we cannot identify any kind of modelling element.

The following is an example from the case of the problem on the safe:

**PF2 - safe**

*The truth is I don't know. I haven't understood this and I wouldn't be able to answer because I have never tried it before.*

In this second example we can see that this student doesn't offer any kind of resolution proposal, admitting he doesn't understand which factors would enable him to draft a plan for the resolution of the problem.

In general, we have come across several types of answers which, as in the examples shown above, do not allow us to find any modelling elements because the students do not try to answer the question asked or do so using erroneous or incoherent reasoning.

### 4.2. Strategies which do not contain modeling

The worksheets which include strategies to answer the posed queries sometimes do not contain any kind of element or process involving modelling.

The following is a clear example of this:

**PB4 - demonstration**

*I would ask some media institution to take an aerial photograph, from a helicopter or a plane, of the whole street the people are occupying.*

In this example we can see that the student passes on the responsibility of estimating the amount of people participating in a demonstration to an external source of information, counting on resources which he actually does not have. We can't find any signs of modelling in this proposal.

Another type of answers in which we don't find modelling processes are those which aim to carry out exhaustive counting, by directly counting objects or people, or by means of a poorly developed recounting method. The following is an example of this:

**PB3 - demonstration**

*Usually, every time a demonstration is held, the organizers leave information slips in people's homes. What I would do is get the people to use this document to confirm their attendance, in order to have a more-or-less approximate number of participants.*

An equivalent option for exhaustive counting of the problem about the swimming pool is the following:

**PE3 - swimming pool**

*I'll take some glasses and start filling them with water from the swimming pool. If I need more glasses I'll get them and keep going until I've used up all of the pool's water. When the swimming pool is empty I'll count the glasses filled and I'll find out how many of them I need.*

In this kind of proposals, the students don't observe some important limitations, such as the time or resources needed to carry out some of the measurements they suggest. In general, we do not observe any kind of scheme which would help them to carry out the counting effectively.
4.3. Strategies which include modelling

The analysis of the data has revealed that some students present suggestions which include modelling elements. These elements differ according to the problem set out and the chosen focus. In fact, we should remember that when asking the students to elaborate only one resolution proposal we will not be able to observe some parts of the complete modelling process. Therefore, the kind of data gathered allows us to identify answers which suggest mathematical models but won't allow us to detect the details on the process of searching for the solution or of its review and validation.

In this study, four general types of strategies have been detected in which modelling processes are observed. These are based on: a) using the rule of product from a previous conception of the arrangement of the elements to be counted; b) introducing a base unit which will be iterated in order to estimate the total number of copies of this unit included in the whole; c) estimating the average concentration of objects to be counted; and d) stratifying the studied population. We expose the main characteristics of each of these strategies as follows.

In the category of those who use the rule of product we have included proposals consistent with recounting the elements with a uniform mental distribution model employing the rule of product. This model may be easily adapted to situations which aim to count the people on a surface area (problems A and B) or the coins in an enclosed space (problem F). This is displayed in the following example:

**PF2 - safe**

I would make a pile of euro-coins until they reach the top, the top lid of the safe (and I would count them). I would put euro-coins horizontally on the bottom of the box until I can’t fit anymore onto it and I would count them. I would multiply one amount by the other and that would be the result.

In the category introducing a base unit, we’ve included the suggestions of pupils who estimate the number of elements using a reference unit, such as the space the average person takes up or the volume of a drop of water. This suggested unit is an abstraction of the concept of the element counted, since, except in recounting text messages or coins, we cannot consider the elements counted to have the same sizes.

The use of this base unit is directly related to the iteration of a unit, which involves applying the mental image of a length unit several times (centimeter, meter...) onto an object to reach an estimate of its length. Several studies have proven that this strategy is used by people who have never worked on other kinds of more elaborate strategies (Hartley 1977, Hildreth 1983).

An improvement on the iteration of a unit would be introducing an everyday object of commonly known dimensions, in order to estimate the length of another object. This strategy is known as iteration of a reference point, in which the reference point is an object which makes a good measurement unit (Carter 1986). Joram, Gabriele, Bertheau, Gelman & Subrahmanyan (2005) have proven that people who make estimates using the iteration of a reference point are significantly more accurate that those who employ the iteration of units. The following is a sample of this:

**PA3 - concert**

I would take 10 students and calculate the space each of them takes up. After that, I would calculate the average of that to find out more or less the number of students that would fit in the playground. I would calculate the total surface area of the whole playground and subtract the meters occupied by the stage. The remaining space, which would be where the people are placed, would be divided by the average space taken up by each student. Therefore, if 108 students fit in the playground, I would sell 100 tickets, because otherwise there will hardly be space to breathe in the playground.
The last category we have identified which contains modelling elements is calculating the concentration of elements. In this category we group the proposals which are based on population density calculations, the number of drops of water in a given volume or the average number of text messages sent per person. The following example depicts this type of answer:

**PC3 - SMS**

Firstly, I would create a spreadsheet with the names of 25 people (between family and friends) and the days of the month. In order to fill in the spreadsheet with the text messages that each person sends in one day of the month, I would give them the sheet and they should commit themselves to fill it in with the number of texts they send. When the month is over, I would collect the results and classify them in one main spreadsheet. I would calculate the average of all the SMS sent by the 25 people. Once I had the average I would multiply it by the total population.

The last type of modelling strategy we detected in the studied resolution proposals is stratifying the population. It has exclusively been used in problem C, which asks for an estimate of text messages sent in one day in a province. In this case, the students suggest grouping the population into different strata which, due to their characteristics, would make a different use of text messaging. Since approaching this type of model requires making distinctions amongst the population being counted, this model isn't found in proposals for other problems. The following is a sample of this kind of proposal:

**PC3 - text messages**

I would carry out a survey by taking one person from each group (differentiating men and women): elderly person (60-70 years old), mature adult (40-60), adult (18-40), teenager (14-18), children (9-14). I would ask each of them how many messages they send every day, supposing all the others would give the same answer, I would multiply that number by the amount of Catalan citizens of his group. I would take the multiplied answers (from all groups, men and women) and add them up. This should give me a very approximate result.

5. Results: Models proposed in the problems

We are able to characterize the models underlying the students' proposals by using the previously identified strategies, presented as follows:

Firstly, we would like to stress that we have proven some students can model the required situation by means of a distribution of elements arranged into rows and columns, that is to say, using the strategy we've called Rule of product. These students suggest counting the number of elements in each row and column in order to then multiply these two values.

We find this kind of models in three different problems. On the one hand, a rectangular distribution is used in problems A and B, which require the estimation of the number of people on a surface area. An example of this would be that exposed in figure 2, which shows a sketch of the playground with the basket to the right of the figure, the area set aside for the stage and the seats, arranged in a rectangular grid-shape.

![Figure 2: Rectangular distribution model - Problem B](image)
Problem F suggests the same kind of model but with a cubic distribution. Figure 3 displays the proposal of a student who suggests calculating the number of coins which fit in each edge of a cube, obtaining two values ($Z=$number of coins which fit into one edge of the base of the cube, $Y=$number of coins which fit into one vertical edge of the cube) to obtain the estimate of coins using the formula $Z^2 \cdot Y$, not shown in the figure.

![Figure 3: Cubic distribution model - Problem F](image)

Another kind of mathematical model the students used is the *iteration of a reference point*, supported by the strategy we've called *introduction of a base unit*, which is based on a strategy for estimating lengths.

It makes sense to characterize the object chosen as a base unit in the analysis of modelling processes which are based on the iteration of a reference point. In this study we have detected that students suggest several reference points to estimate the number of people/objects which make up a whole using a relevant property. For instance, in the case of counting the people occupying a given space, the surface a person takes up is established as a reference point. In order to determine a value for the number of people, the total space is divided by the space one person takes up. The reference points we reported are the following:

- **Chair per person:** Average surface occupied by a person seated on a chair. Detected in problem A.
- **Average person:** Average surface occupied by a person standing up. Detected in problems A and B.
- **Weight of a drop:** Average weight of a drop of water. Detected in problem D.
- **Volume of a glass:** Average volume of a glass. Detected in problem E.
- **Volume of a drop:** Average volume of a drop of water. Detected in problem D.
- **Volume of a coin:** Average volume of a coin. Detected in problem F.

The third kind of model that appears in the resolution proposals is supported by the strategy we called *calculating the concentration of elements* in which the students use *concentration measurements*. By concentration measurements we understand the numerical proportion which reflects the relationship between two magnitudes. One of the examples detected in the students' proposals is population density, which the students know from other school subjects. However, we have also detected other measurements of concentration which the pupils suggest, generated specifically for each kind of problem.
The different concentration measurements detected for each problem are the following:

- **Population density**: Average number of people in one square meter. Detected in problems A and B.
- **Average of text messages**: Average number of text messages sent by one person in one day. Detected in problem C.
- **Glasses in a volume of water**: Average number of glasses required to fill up a certain amount of water. Detected in problem E.
- **Water in a given time**: Average amount of water collected in a container in a certain amount of time. Detected in problem D.

The last kind of model the students proposed is the *stratification of the population*. This type of model has only been detected in problem C, in which the students aim to separate the population according to the use they make of text messaging. In this way, the only reported model is the following:

- **Stratification of the population by age**: The studied population is separated into different age groups in order to assign an average number of text messages sent to each group. Detected in problem C.

If we consider the presence of modelling elements in each of the problems quantitatively, we obtain the relation which appears in table 3.

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<th>PA</th>
<th>PB</th>
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<td>25</td>
<td>51</td>
<td>54</td>
<td>57</td>
<td>35</td>
<td>262</td>
</tr>
<tr>
<td>Total</td>
<td>96</td>
<td>92</td>
<td>90</td>
<td>88</td>
<td>86</td>
<td>86</td>
<td>538</td>
</tr>
</tbody>
</table>

Table 3: Distribution of proposals by level of modelling and problem

As you can observe in the table, a great deal of the proposals which appear in the students' worksheets don't contain modelling elements. However, we consider that the proportion of samples which include generating models is high, since nearly half of the proposals suggest models for a type of problems which hadn't ever been worked on with the students.

The strategies with modelling elements in them make up 49% of the total. The problems with the highest percentage of proposals which include modelling are problem E, 66%, and problem D, 61%. These two problems have an equivalent mathematical approach but are set in two different situations. On the other hand, problems A and B, which are also mathematically equivalent in different contexts, show great differences in the percentage of proposals with modelling elements. In problem A we find 42% and just 27% in problem B. We interpret this fact as related to the differences between the contexts suggested in problems A and B. They both refer to counting the number of people on a surface area, but the surface of the school playground (problem A) is closed, constant and known by the students, whereas that of the demonstration (problem B) hasn't been specified, doesn't have physical borders and isn't fixed. However, problems D and E are presented in similar contexts but their differences lie in the type of model suggested (table 4), since the students are more familiar with the volume of a glass and use it as a base unit more frequently than the volume of a drop of water.

This suggests that it is also important to quantitatively analyze what kind of strategies appear in the different problems when the students make proposals which include modelling elements. Table 4 relates this, displaying the number of proposals detected for each model and problem.

<table>
<thead>
<tr>
<th></th>
<th>PA</th>
<th>PB</th>
<th>PC</th>
<th>PD</th>
<th>PE</th>
<th>PF</th>
<th>Total</th>
</tr>
</thead>
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<td>Rule of the product</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td>Unit iteration</td>
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<td>0</td>
<td>24</td>
<td>52</td>
<td>25</td>
<td>128</td>
</tr>
<tr>
<td>Concentration measurement</td>
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<td>16</td>
<td>45</td>
<td>30</td>
<td>5</td>
<td>2</td>
<td>113</td>
</tr>
<tr>
<td>Stratification</td>
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<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>40</td>
<td>25</td>
<td>51</td>
<td>54</td>
<td>57</td>
<td>35</td>
<td>262</td>
</tr>
</tbody>
</table>
Table 4: distribution of proposals by modelling strategy and problem

We can see that some of the models only take place in some specific problems, although we find suggestions which are focused on concentration measurements in all of the problems. Noteworthy is the fact that problems equivalent in their approach (such as problems A and B, which require estimating the number of people on a given surface) induce the students to use different models according to the situation arisen.

It is also interesting to analyze the possible relation between the students' age and the presence of modelling elements in their proposals. Table 5 shows, for each year, the number of detected proposals which contain modelling elements and the percentage of the total they represent, over the total number of students in each year. An increase in the percentage of modelling proposals is observed as the level of the students increases. Thus, this is more related to maturity and understanding of the environment than to the studied mathematical concepts, since we find valid proposals with similar reasoning in all levels. However, we have also found proposals which don't offer any kind of resolution strategy.

<table>
<thead>
<tr>
<th>Modelling proposals</th>
<th>Total</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st ESO</td>
<td>39</td>
<td>111</td>
</tr>
<tr>
<td>2nd ESO</td>
<td>50</td>
<td>127</td>
</tr>
<tr>
<td>3rd ESO</td>
<td>69</td>
<td>133</td>
</tr>
<tr>
<td>4th ESO</td>
<td>99</td>
<td>167</td>
</tr>
</tbody>
</table>

Table 5: Percentage of proposals which model the situation by school years

6. Discussion and conclusions

In this study we have observed that some students are able to generate mathematical models which will allow them to obtain the estimates the problems seek. However, a large part of the pupils does not offer any action plans with valid strategies, which could be carried out within a reasonable period of time or with the resources at hand. Consequently, our results demonstrate that not all students are competent enough at problem-solving, at least in this kind of problems. We have nonetheless observed that some students will suggest strategies which model a situation when the right kind of problems is chosen. Following Pólya, who supports the idea of the question ¿do you know any related problems? as the educational driving force for problem-solving, we believe that using sequences of problems, such as those presented herein, in situations with developing complexities could be one way of introducing modelling in secondary school.

The results of our study lead us to conclude that secondary school students have enough mathematical knowledge to address Fermi Problems involving inconceivable magnitude estimation and can attempt to solve them by employing modelling processes. In fact, we have detected several different types of models suggested by the students for just one problem. The abundance of possible strategies may trigger interesting discussions in the classroom. On the other hand, some kinds of models have been detected in more than one problem, thus Fermi problems centered on inconceivable magnitude estimation may be a means of generalizing mathematical models to other situations (English 2006, Doerr & English 2003).

The whole resolution process has not been observed in this study and since modelling processes are multi-cyclical (Borromeo Ferri 2006) we can only infer the students' behaviour during the whole problem-solving process. However, for some of the problems dealt with, the students were later asked to complete the resolution in class working in small groups. The development of the collected proposals suggests that the practical limitations students come across when attempting to carry out their initial plans could speed up the modelling process required for the effectiveness of the resolution process.
We therefore support that Fermi problems based on inconceivable magnitude estimation represent an interesting option to introduce mathematical modelling in secondary school. Backed up by the obtained data, we expect that if a large part of students are able to model this kind of situations, we can get model creation to become a suitable activity for secondary school classrooms. Considering this, project teamwork may lead a higher number of students to take part in modelling processes.

References


