Torsional wave in a homogeneous layer over a heterogeneous half-space - A Mathematical Model

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Abstract
The paper aims to study the propagation of torsional surface waves in a homogeneous layer over a semi infinite heterogeneous half space. The heterogeneity has been considered both in rigidity and density. The study reveals that under assumed conditions, a torsional surface wave propagates in the medium. The velocities of torsional surface waves have been obtained. It is also observed that for a layer over a homogeneous half-space, the velocity of torsional surface waves coincides with that of Love waves. An attempt is also made to assess the possible propagation of torsional surface waves in a half-space with polynomial variation in rigidity and density, lacking a superficial layer.

Key words: torsional surface waves; propagation; heterogeneous half-space, rigidity.

1. Introduction

Due to non availability of sufficient literatures in connection with the propagation of torsional surface waves, this wave has practically ignored in the study of seismic wave propagation. In seismogram some disturbance are observed in between the arrival of Rayleigh wave and Love wave disturbance. As sufficient informations were not available these disturbances termed as “noise” and are ignored in the study of seismic waves. These noises may be due to the torsional wave which propagates in the non homogeneous earth and the attention of the seismologist may be drawn. That motivates us to study the propagation of Torsional wave in this paper. This wave is one type of surface wave. It gives twist to the medium during the propagation of earthquake. Surface waves propagating over the surface of homogeneous and inhomogeneous elastic half spaces are a well known and prominent feature of wave theory.

Quite a good amount of information about the propagation of seismic waves is contained in the well-known book by Ewing et al. [1]. Numerous papers on the subject have been published in various journals. In fact, the study of surface waves for homogeneous, non-homogeneous and layered media has been a central interest to theoretical seismologists until recently. Of these, the commendable works by Vrettose [2, 3] on surface waves in inhomogeneous medium may be cited. His study gives so much insight on the effects of non-homogeneity on surface waves causes by line loads, although much information is available on the propagation of surface waves, such as Rayleigh waves, Love waves and Stoneley waves, etc., torsional waves have not drawn much attention and only scanty literature is available on the propagation of such waves. Rayleigh [4] has shown that an isotropic homogeneous elastic half-space does not allow torsional surface waves to propagate Later on, Meissner[5] pointed out that in an inhomogeneous elastic half-space with quadratic variation of shear modulus and density varying linearly with depth, torsional surface waves do exist. Recently, Vardoulakis [6] has shown that torsional surface wave also propagate in a Gibson half-space, that is a half-space in which the shear modulus varies linearly with depth but the density remains unchanged. Georgiadis et al [7] has shown...
that torsional surface wave do exists in gradient elastic half space. Torsional waves in an initially
stressed cylinder have been studied by Dey and Dutta[8] and the existence and propagation of
torsional surface waves in an elastic half-space with void pores has been discussed by Dey et al[9].

This paper presents a study on the propagation of torsional surface waves in a homogeneous
layer of finite thickness over a heterogeneous half-space. The density and rigidity of the Earth varies
with depth. Both rigidity and density of the half-space are assumed to vary polynomially with depth.

2. Formulation

\[
\begin{array}{c|c}
Z=-H & \mu = \mu_0 \\
\hline
Z = 0 & \mu = \mu_i \left(1+az\right)^n \\
\end{array}
\]

\[
\begin{array}{c|c}
Z = 0 & \rho = \rho_0 \\
\hline
Z = 0 & \rho = \rho_i \left(1+az\right)^n \\
\end{array}
\]

\[\text{Figure 1: Geometry of the problem}\]

Consider a homogeneous layer of thickness H, over a vertically heterogeneous half-space. The
heterogeneity has been considered both in density and rigidity.

Considering the origin of the cylindrical co-ordinate system at the interface of the layer and
the z-axis downward positive, the following variation in rigidity and density is taken:

(i) for the layer, \( \mu = \mu_0, \quad \rho = \rho_0 \)

(ii) for the half space, \( \mu = \mu_i \left(1+az\right)^n, \quad \rho = \rho_i \left(1+az\right)^n \)

where \( \mu \) and \( \rho \) are rigidity and density of the media, respectively, and \( a \) is constant having
dimension that is inverse of length.

Equation of motion

The dynamical equations of motion for the system are given as

\[
\begin{align*}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \sigma_{rr} - \sigma_{\theta \theta} &= \rho \frac{\partial^2 u}{\partial t^2}, \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + 2\sigma_{r \theta} &= \rho \frac{\partial^2 v}{\partial t^2}, \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r \theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \sigma_{rz} &= \rho \frac{\partial^2 w}{\partial t^2},
\end{align*}
\]

Where \( \rho \) is the density, \( \sigma_{rr}, \sigma_{\theta \theta}, \sigma_{\theta z}, \sigma_{rz}, \sigma_{r \theta}, \) and \( \sigma_{\theta z} \) are the corresponding stress
components in their conventional sense, \( u, v, w \) are the displacement components in radial ,
circumferential and axial directions.
In cylindrical co-ordinate the stress-strain relations are taken as
\[
\begin{align*}
\sigma_{rr} &= \lambda \Omega + 2\mu e_{rr} & \sigma_{\theta\theta} &= \lambda \Omega + 2\mu e_{\theta\theta} \\
\sigma_{zz} &= \lambda \Omega + 2\mu e_{zz} & \sigma_{r\theta} &= 2\mu e_{r\theta} \\
\sigma_{\theta z} &= 2\mu e_{\theta z} & \sigma_{\theta z} &= 2\mu e_{\theta z}
\end{align*}
\] (3)

Where \( \lambda \) and \( \mu \) are Lame's coefficient and

Strain displacement relations are given by

\[
\begin{align*}
e_{rr} &= \frac{\partial u}{\partial r} & e_{\theta\theta} &= \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \\
e_{zz} &= \frac{\partial w}{\partial z} & e_{r\theta} &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \\
e_{\theta z} &= \frac{\partial v}{\partial \theta} + \frac{1}{r} \frac{\partial w}{\partial r} & e_{zr} &= \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}
\end{align*}
\] (4)

and cubical dilatation \( \Omega = \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u + \frac{\partial w}{\partial r}}{r} \right) \)

For the present problem we are interested to study the propagation of torsional surface wave in the medium. Following the usual method for the problems having \( \theta \) symmetry it can easily be seen that 1\textsuperscript{st} and 3\textsuperscript{rd} equation of (2) are automatically satisfied as \( u = 0 \) and \( w = 0 \).

Hence for torsional wave propagation in the radial direction, the equation of motion may be written as

\[
\frac{\partial}{\partial r} \sigma_{rr} + \frac{\partial}{\partial z} \sigma_{zz} + \frac{2}{r} \sigma_{r\theta} = \rho(z) \frac{\partial^2 v}{\partial t^2}
\] (5)

with \( v(r, z, t) \) being the displacement along the \( \theta \) (azimuthal) direction. For an elastic medium the stresses are related to the displacement component \( v \) by

\[
\begin{align*}
\sigma_{r\theta} &= \mu(z) \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right) \\
\sigma_{\theta z} &= \mu(z) \left( \frac{\partial v}{\partial z} \right)
\end{align*}
\] (6)

Using eq. (6), eq. (5) takes the form

\[
\mu(z) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) v + \frac{\partial}{\partial z} \left( \mu(z) \frac{\partial v}{\partial z} \right) = \rho(z) \frac{\partial^2 v}{\partial t^2}
\] (7)

We assume a solution of eq.(7) of the form

\[
v = V(z) J_1(Kr)e^{iat}
\] (8)

where \( V(z) \) is the solution to

\[
V''(z) + \frac{\mu'(z)}{\mu(z)} V'(z) - K^2 \left( 1 - \frac{c^2}{c_s^2} \right) V(z) = 0
\] (9)

in which \( c = \frac{\omega}{K} \) and \( c_s = \sqrt{\frac{\mu}{\rho}} \) and \( J_1(Kr) \) is the Bessel function of first kind.

Solution for the upper layer

For the upper layer \( \mu = \mu_o \) and \( \rho = \rho_o \)

(10)
Using eq. (10) and eq. (9), for the layer takes the form

\[ V''(z) - K^2 \left( 1 - \frac{c^2 V(z)}{c_0^2} \right) V(z) = 0 \]  

(11)

where

\[ c_0 = \sqrt{\frac{\mu_0}{\rho_0}} \]  

(12)

The solution of eq. (11) is

\[ V(z) = A_1 \exp \left( 1 - \frac{c^2 V(z)}{c_0^2} \right) Kz + A_2 \exp \left( 1 - \frac{c^2 V(z)}{c_0^2} \right) Kz \]  

(13)

and hence the displacement in the homogeneous layer is

\[ v_0 = \left[ A_1 \exp \left( 1 - \frac{c^2 V(z)}{c_0^2} \right) Kz + A_2 \exp \left( 1 - \frac{c^2 V(z)}{c_0^2} \right) Kz \right] J_1(Kr)e^{i\alpha z} \]  

(14)

**Solution for the half-space**

For the half-space \( \mu = \mu_1 (1 + az)^n \) and \( \rho = \rho_1 (1 + az)^n \)

Using eq. (15), eq. (9) takes the form

\[ V''(z) + \frac{2na}{1 + az} V'(z) - K^2 \left( 1 - \frac{c^2 (1 + az)^n}{c_1^2} \right) V(z) = 0 \]  

(16)

where \( c_1 = \sqrt{\frac{\mu_1}{\rho_1}} \)

\[ V''(z) + \frac{2na}{1 + az} V'(z) - K^2 \left( 1 - \frac{c^2 (1 + az)^n}{c_1^2} \right) V(z) = 0 \]  

(17)

Substituting \( V(z) = \frac{\phi(z)}{(1 + az)^n} \) and \( m_z^2 = \left( 1 - \frac{c^2}{c_1^2} \right) \) in eq. (17), we have

\[ \phi''(z) + K^2 m_z^2 \frac{q}{(1 + az)^2 K^2 m_z^2 - 1} \phi(z) = 0 \]  

(18)

Where \( q = -\frac{n}{2} \left( \frac{n}{2} - 1 \right) a^2 \)

Using dimensionless quantities \( b = \frac{Km_z}{a} \), \( p = \frac{b^2 q}{K^2 m_z^2} \) and \( \eta = 2(Km_z + b) \) in eq. (18), we obtain

\[ \phi''(\eta) + \left[ -\frac{1}{4} + \frac{p}{\eta^2} \right] \phi(\eta) = 0 \]  

(19)
The solution of eq. (19) satisfying the condition \( \lim_{z \to 0} V(z) \to 0 \) i.e., \( \lim_{\eta \to \infty} \phi(\eta) \to 0 \) may be taken as

\[
\phi(\eta) = DW_{0,H_1}(\eta)
\]

where \( W_{0,H_1}(\eta) \) is the Whittaker function and \( H_1^2 = \frac{1}{4} - p \)

Hence, the displacement component \( v \) in the heterogeneous half space is given by

\[
v_1 = \frac{DW_{0,H_1} \left[ 2 \left( Km_2 z + b \right) \right]}{(1 + az)^{\frac{n}{2}}} J_1(Kr) e^{iax}
\]

(20)

**Boundary conditions**

The appropriate boundary conditions are as follows.

(a) At the upper boundary \( z = -H \)

stress component

\[
\mu_0 \left( \frac{\partial v_0}{\partial z} \right) = 0
\]

(21)

(b) At the interface \( z = 0 \)

(i) continuity of stress component

\[
\mu_0 \left( \frac{\partial v_0}{\partial z} \right) = \mu_1 \left( \frac{\partial v_1}{\partial z} \right)
\]

(ii) continuity of the displacement component

\[
v_0 = v_1
\]

(22)

using the boundary condition of (21) eq.(14) takes the form

\[
A_1 \exp \left\{ -\left(1 - \frac{c^2}{c_0^2}\right)^{\frac{1}{2}} KH \right\} - A_2 \exp \left\{ \left(1 - \frac{c^2}{c_0^2}\right)^{\frac{1}{2}} KH \right\} = 0
\]

(23)

Expanding Whittaker's function up to linear terms and substituting into the boundary conditions of (22), we get

\[
A_1 + A_2 = D \left(2b\right)^{\frac{1}{2} + H_1} e^{-b} \left[1 + b\right]
\]

(24)

and

\[
\mu_0 K \left[ \frac{1 - \frac{c^2}{c_0^2}}{2} \right] \left( A_1 - A_2 \right) = D \mu_1 \left(2b\right)^{\frac{1}{2} + H_1} e^{-b} \left(1 + b\right) \left[ \left(\frac{1}{2} + H_1\right) a - Km_2 \frac{b}{1 + b} - \frac{n}{2} a \right]
\]

(25)

Eliminating \( A_1, A_2 \) and \( D \) from eq. (23), eq. (24) and eq. (25) we obtain

\[
\frac{\mu_1}{\mu_0} \left[ \frac{\frac{m_2}{K} + \frac{m_2}{\frac{n}{2}}}{a} \right] = \tan \left\{ KH \left( \frac{c^2 - 1}{c_0^2 - 1} \right) \right\}
\]

(26)

This gives the velocity of torsional surface waves in a homogeneous layer over a vertically heterogeneous half-space.
Particular cases

Case I. When \( a \to 0 \), i.e. the layer and the half-space have both constant density and rigidity, the eq. (26), we obtain

\[
\frac{\mu_1}{\mu_0} \left( \frac{1 - \frac{c^2}{c_1^2}}{\frac{c^2}{c_0^2} - 1} \right)^{1/2} = \tan \left( KH \left( \frac{c^2}{c_0^2} - 1 \right)^{1/2} \right)
\]

This is the well-known equation of Love waves in a homogeneous layer over a homogeneous half-space. This shows that in a layered isotropic homogeneous medium, torsional surface waves change asymptotically to Love waves as the distance from the origin increases.

Case II. In the absence of the upper layer (\( H \to 0 \)) eq. (26) then takes the form

\[
\left[ \frac{m^2}{a + m^2} - \left( \frac{1}{2} + H_1 - \frac{n}{2} \right) \frac{a}{K} \right] = 0
\]

Eq. (28) takes the form

\[
c^2 + X_1c + X_2 = 0
\]

Where

\[
c = \left( \frac{c}{c_1} \right)^2
\]

\[
X_2 = 1 - \left( \frac{a}{K} \right)^2 \left( 2P + P^2 \right) + \left( \frac{a}{K} \right)^4 P^2
\]

\[
X_1 = -2 + \left( \frac{a}{K} \right)^2 \left( 2P + P^2 \right)
\]

\[
P = \frac{1}{2} + H_1 - \frac{n}{2}
\]

3. Example of applications and discussion of the results

The values of \( c / c_0 \) have been computed from eq. (26) for \( a / K = 0.01 \) and \( \mu_0 / \mu_1 = 0.5 \) and for different values of \( KH \). Here we plotted Fig. 2 and Fig. 3 by taking \( \left( c_0 / c_1 \right)^2 = 0.2 \) and 0.4 respectively. The numerical values of \( c / c_0 \) have been calculated from eq. (27) for \( \mu_0 / \mu_1 = 0.5 \) i.e. the velocity of Love wave in homogeneous layer over a homogeneous medium. These values are shown in Fig. 4 for \( \left( c_0 / c_1 \right)^2 = 0.2 \). Where as Fig. 5 shows the same thing for \( \left( c_0 / c_1 \right)^2 = 0.4 \).
Figure 2: Curve of $c/c_0$ versus $KH$ from eq. (26) for $(c_0/c_1)^2 = 0.2$

Figure 3: Curve of $c/c_0$ versus $KH$ from eq. (26) for $(c_0/c_1)^2 = 0.4$

Figure 4: Curve of $c/c_0$ versus $KH$ from eq. (27) for $(c_0/c_1)^2 = 0.2$
4. Conclusions

It is observed that such a medium allows torsional waves to propagate. Prominent effect of variation of density on the propagation of torsional wave is observed in the low frequency range. As a particular case, the velocity of torsional surface waves in a non-homogeneous half-space in the absence of superficial layer is also discussed, and it is found that two torsional wave fronts are possible in the low frequency range. For a homogeneous layer over a homogeneous half-space, it is observed that the speed of torsional waves coincides with that of Love waves.

References


